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Vacuum mimicking phenomena in neutrino oscillations

Osamu Yasuda

Department of Physics, Tokyo Metropolitan University, Minami-Osawa, Hachioji, Tokyo 192-0397, Japan

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Abstract

It is shown generally that any oscillation probability in matter with approximately constant density coincides with that in vacuum to the first two nontrivial orders in $\Delta m^2_{jk}L/E$ if $|\Delta m^2_{jk}L/E| \ll 1$ and $|G_F N_e L| \ll 1$ are satisfied.

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Recently a lot of efforts have been made on study of neutrino oscillations at long baseline experiments. Using the mass hierarchical condition $|\Delta m^2_{21}| \ll |\Delta m^2_{32}| \simeq |\Delta m^2_{31}|$ in the three flavor framework of neutrino oscillations, it has been found in the case of T-conserving probability $P(\nu_e \to \nu_\mu)$ [1–3] or in the case of T-violating probability $P(\nu_\mu \to \nu_e)$ [4,5] that the oscillation probability $P(\nu_\alpha \to \nu_\beta)_{\text{matter}}$ in matter coincides with that $P(\nu_\alpha \to \nu_\beta)_{\text{vacuum}}$ in vacuum

$$P(\nu_\alpha \to \nu_\beta)_{\text{matter}} \simeq P(\nu_\alpha \to \nu_\beta)_{\text{vacuum}}, \tag{1}$$

when $|\Delta m^2_{jk}L/E| \ll 1$ and $|AL| \ll 1$ are satisfied, where $A \equiv \sqrt{2}\,G_F N_e$ stands for the matter effect [6,7] and N_e is the density of electrons. This phenomenon was referred to as vacuum mimicking in [5]. In this short note it is shown that (1) holds in the first two nontrivial orders in $\Delta m^2_{jk}L/2E$ and AL (the terms quadratic and cubic in $\Delta m^2_{jk}L/2E$ correspond to T-conserving and T-violating probabilities in the leading order, respectively) for arbitrary numbers N of neutrino flavors with general form $\text{diag}(A_1, A_2, \ldots, A_N)$ of the matter effect if $|\Delta m_{jk}L/2E| \ll 1$ and $|AL| \ll 1$ are satisfied.

In the three flavor framework of neutrino oscillations, the positive energy part of the Dirac equation which describes neutrino propagation is given by

$$i\frac{d\Psi}{dt} = \left[U\,\text{diag}(E_1, E_2, E_3)U^{-1} + \text{diag}(A, 0, 0)\right]\Psi, \tag{2}$$

E-mail address: yasuda@phys.metro-u.ac.jp (O. Yasuda).

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where $\Psi^T \equiv (\nu_e, \nu_\mu, \nu_\tau)$ is the flavor eigenstate, U is the Pontecorvo–Maki–Nakagawa–Sakata [8–10] (PMNS) matrix,[1] and $E_j \equiv \sqrt{m_j^2 + \vec{p}^2}$. Throughout this paper we assume that the density of matter is constant for simplicity. The case of matter with slowly varying density will be briefly discussed at the end of the Letter.

Here let us consider more general case with N neutrino flavors and with general matter effect:

$$i\frac{d\Psi}{dt} = \bigl(U\mathcal{E}U^{-1} + \mathcal{A}\bigr)\Psi, \tag{3}$$

where

$$\mathcal{E} \equiv \mathrm{diag}(E_1, E_2, \ldots, E_N), \tag{4}$$

$$\mathcal{A} \equiv \mathrm{diag}(A_1, A_2, \ldots, A_N), \tag{5}$$

U is the $N \times N$ PMNS matrix and $\Psi^T \equiv (\nu_{\alpha_1}, \nu_{\alpha_2}, \ldots, \nu_{\alpha_N})$ is the flavor eigenstate. Without the matter effect (i.e., $A_j = 0,\ j = 1, \ldots, N$), (3) can be easily solved and the oscillation probability $P(\nu_\alpha \to \nu_\beta)_{\text{vacuum}}$ is given by

$$P(\nu_\alpha \to \nu_\beta)_{\text{vacuum}} = \delta_{\alpha\beta} - 2\sum_{j,k} U_{\alpha j}U^*_{\beta j}U^*_{\alpha k}U_{\beta k}\sin^2\left(\frac{\Delta E_{jk}L}{2}\right) - i\sum_{j,k} U_{\alpha j}U^*_{\beta j}U^*_{\alpha k}U_{\beta k}\sin\bigl(\Delta E_{jk}L\bigr), \tag{6}$$

where $\Delta E_{jk} \equiv E_j - E_k$ and the second and the third terms on the right hand side correspond to CP-conserving and CP-violating probabilities, respectively.

With the nonvanishing matter effect, on the other hand, explicit evaluation of the probability is difficult but the $N \times N$ matrix $U\mathcal{E}U^{-1} + \mathcal{A}$ on the right hand side of (3) can be formally diagonalized by an $N \times N$ unitary matrix U^M:

$$U\mathcal{E}U^{-1} + \mathcal{A} = U^M\mathcal{E}^M\bigl(U^M\bigr)^{-1}, \tag{7}$$

where

$$\mathcal{E}^M \equiv \mathrm{diag}\bigl(E_1^M, E_2^M, \ldots, E_N^M\bigr), \tag{8}$$

and E_j^M stands for the eigenvalue of $U\mathcal{E}U^{-1} + \mathcal{A}$. As in the case of the oscillation probability in vacuum, we can formally solve (3) and express the oscillation probability $P(\nu_\alpha \to \nu_\beta)_{\text{matter}}$ as

$$P(\nu_\alpha \to \nu_\beta)_{\text{matter}} = \delta_{\alpha\beta} - 2\sum_{j,k} U^M_{\alpha j}U^{M*}_{\beta j}U^{M*}_{\alpha k}U^M_{\beta k}\sin^2\left(\frac{\Delta E^M_{jk}L}{2}\right) - i\sum_{j,k} U^M_{\alpha j}U^{M*}_{\beta j}U^{M*}_{\alpha k}U^M_{\beta k}\sin\bigl(\Delta E^M_{jk}L\bigr), \tag{9}$$

where $\Delta E^M_{jk} \equiv E^M_j - E^M_k$ and the second and the third terms on the right hand side correspond to T-conserving and T-violating probabilities, respectively.

Now let us assume that $|\Delta E_{jk}L| \ll 1$ and $|\Delta E^M_{jk}L| \ll 1$ are satisfied, where the latter follows if $|\Delta E_{jk}L| \ll 1$ and $|A_jL| \ll 1$. Then we can expand the sine functions in (6) and (9). The zeroth order term is obviously $\delta_{\alpha\beta}$ for both probabilities. The term linear in $\Delta E_{jk}L$ vanishes, since

$$\begin{aligned}\sum_{j,k} U_{\alpha j}U^*_{\beta j}U^*_{\alpha k}U_{\beta k}\Delta E_{jk}L &= L\sum_{j,k} U_{\alpha j}U^*_{\beta j}U^*_{\alpha k}U_{\beta k}(E_j - E_k)\\ &= L\left[\delta_{\alpha\beta}\sum_j U_{\alpha j}U^*_{\beta j}E_j - \delta_{\alpha\beta}\sum_k U^*_{\alpha k}U_{\beta k}E_k\right] = L\delta_{\alpha\beta}\Bigl[\bigl(U\mathcal{E}U^{-1}\bigr)_{\alpha\beta} - \bigl(U\mathcal{E}U^{-1}\bigr)_{\beta\alpha}\Bigr] = 0,\end{aligned} \tag{10}$$

[1] Following S.T. Petcov [11], we call U the PMNS matrix.

where $\delta_{\alpha\beta}$ has been obtained from the unitarity condition $\sum_j U_{\alpha j}U^*_{\beta j}=\delta_{\alpha\beta}$, and the last equality holds because the inside of the square bracket vanishes for $\alpha=\beta$. Similarly, we have

$$\sum_{j,k} U^M_{\alpha j}U^{M*}_{\beta j}U^{M*}_{\alpha k}U^M_{\beta k}\Delta E^M_{jk}L = L\delta_{\alpha\beta}\Big[\Big(U^M\mathcal{E}\big(U^M\big)^{-1}\Big)_{\alpha\beta} - \Big(U^M\mathcal{E}\big(U^M\big)^{-1}\Big)_{\beta\alpha}\Big] = 0. \tag{11}$$

The first nontrivial case is the term quadratic in $\Delta E_{jk}L$ and $\Delta E^M_{jk}L$. From (9) we have the term quadratic in $\Delta E^M_{jk}L$ (up to a factor $-1/2$)

$$\begin{aligned}\sum_{j,k} U^M_{\alpha j}U^{M*}_{\beta j}U^{M*}_{\alpha k}U^M_{\beta k}\big(\Delta E^M_{jk}L\big)^2 &= L^2\sum_{j,k} U^M_{\alpha j}U^{M*}_{\beta j}U^{M*}_{\alpha k}U^M_{\beta k}\Big[\big(E^M_j\big)^2 - 2E^M_jE^M_k + \big(E^M_k\big)^2\Big]\\ &= L^2\Big[\delta_{\alpha\beta}\sum_j U^M_{\alpha j}U^{M*}_{\beta j}\big(E^M_j\big)^2 + \delta_{\alpha\beta}\sum_k U^{M*}_{\alpha k}U^M_{\beta k}\big(E^M_k\big)^2 - 2\sum_j U^M_{\alpha j}U^{M*}_{\beta j}E^M_j\sum_k U^{M*}_{\alpha k}U^M_{\beta k}E^M_k\Big]. \end{aligned}\tag{12}$$

Here we note the following properties:

$$\sum_j U^M_{\alpha j}U^{M*}_{\beta j}E^M_j = \big(U\mathcal{E}U^{-1}+\mathcal{A}\big)_{\alpha\beta} = \big(U\mathcal{E}U^{-1}\big)_{\alpha\beta} + \delta_{\alpha\beta}\mathcal{A}_\alpha, \tag{13}$$

$$\begin{aligned}\sum_j U^M_{\alpha j}U^{M*}_{\beta j}\big(E^M_j\big)^2 &= \Big[U^M\big(\mathcal{E}^M\big)^2\big(U^M\big)^{-1}\Big]_{\alpha\beta} = \Big\{\big[U^M\mathcal{E}^M\big(U^M\big)^{-1}\big]^2\Big\}_{\alpha\beta} = \Big[\big(U\mathcal{E}U^{-1}+\mathcal{A}\big)^2\Big]_{\alpha\beta}\\ &= \big(U\mathcal{E}^2U^{-1}\big)_{\alpha\beta} + (\mathcal{A}_\alpha+\mathcal{A}_\beta)\big(U\mathcal{E}U^{-1}\big)_{\alpha\beta} + \delta_{\alpha\beta}(\mathcal{A}_\alpha)^2. \end{aligned}\tag{14}$$

Thus (12) becomes

$$\begin{aligned}& L^2\delta_{\alpha\beta}\Big\{\Big[U^M\big(\mathcal{E}^M\big)^2\big(U^M\big)^{-1}\Big]_{\alpha\beta} + \Big[U^M\big(\mathcal{E}^M\big)^2\big(U^M\big)^{-1}\Big]_{\beta\alpha}\Big\}\\ &\quad - 2L^2\Big[U^M\mathcal{E}^M\big(U^M\big)^{-1}\Big]_{\alpha\beta}\Big[U^M\mathcal{E}^M\big(U^M\big)^{-1}\Big]_{\beta\alpha}\\ &\quad = 2L^2\delta_{\alpha\beta}\Big[\big(U\mathcal{E}^2U^{-1}\big)_{\alpha\alpha} + 2\mathcal{A}_\alpha\big(U\mathcal{E}U^{-1}\big)_{\alpha\alpha} + (\mathcal{A}_\alpha)^2\Big]\\ &\qquad - 2L^2\Big[\big(U\mathcal{E}U^{-1}\big)_{\alpha\beta} + \delta_{\alpha\beta}\mathcal{A}_\alpha\Big]\Big[\big(U\mathcal{E}U^{-1}\big)_{\beta\alpha} + \delta_{\alpha\beta}\mathcal{A}_\alpha\Big]\\ &\quad = 2L^2\Big[\delta_{\alpha\beta}\big(U\mathcal{E}^2U^{-1}\big)_{\alpha\alpha} - \big(U\mathcal{E}U^{-1}\big)_{\alpha\beta}\big(U\mathcal{E}U^{-1}\big)_{\beta\alpha}\Big], \end{aligned}\tag{15}$$

where all the contributions of the matter effect have disappeared in the last step. Since the last expression in (15) is the term quadratic in $\Delta E_{jk}L$ for the probability in vacuum, we obtain

$$\sum_{j,k} U^M_{\alpha j}U^{M*}_{\beta j}U^{M*}_{\alpha k}U^M_{\beta k}\big(\Delta E^M_{jk}L\big)^2 = \sum_{j,k} U_{\alpha j}U^*_{\beta j}U^*_{\alpha k}U_{\beta k}\big(\Delta E_{jk}L\big)^2. \tag{16}$$

Next let us turn to the term cubic in $\Delta E^M_{jk}L$. It is given by (up to a factor $i/3!$)

$$\begin{aligned}&\sum_{j,k} U^M_{\alpha j}U^{M*}_{\beta j}U^{M*}_{\alpha k}U^M_{\beta k}\big(\Delta E^M_{jk}L\big)^3\\ &\quad = L^3\sum_{j,k} U^M_{\alpha j}U^{M*}_{\beta j}U^{M*}_{\alpha k}U^M_{\beta k}\Big[\big(E^M_j\big)^3 - 3\big(E^M_j\big)^2E^M_k + 3E^M_j\big(E^M_k\big)^2 - \big(E^M_k\big)^3\Big]\\ &\quad = L^3\delta_{\alpha\beta}\Big\{\Big[U^M\big(\mathcal{E}^M\big)^3\big(U^M\big)^{-1}\Big]_{\alpha\beta} - \Big[U^M\big(\mathcal{E}^M\big)^3\big(U^M\big)^{-1}\Big]_{\beta\alpha}\Big\}\end{aligned}$$

$$
\begin{aligned}
&- 3L^3\Big[U^M\big(\mathcal{E}^M\big)^2\big(U^M\big)^{-1}\Big]_{\alpha\beta}\Big[U^M\mathcal{E}^M\big(U^M\big)^{-1}\Big]_{\beta\alpha} \\
&+ 3L^3\Big[U^M\mathcal{E}^M\big(U^M\big)^{-1}\Big]_{\alpha\beta}\Big[U^M\big(\mathcal{E}^M\big)^2\big(U^M\big)^{-1}\Big]_{\beta\alpha} \\
= &-3L^3\Big[\big(U\mathcal{E}^2U^{-1}\big)_{\alpha\beta} + (\mathcal{A}_\alpha + \mathcal{A}_\beta)\big(U\mathcal{E}U^{-1}\big)_{\alpha\beta} + \delta_{\alpha\beta}(\mathcal{A}_\alpha)^2\Big]\Big[\big(U\mathcal{E}U^{-1}\big)_{\beta\alpha} + \delta_{\alpha\beta}\mathcal{A}_\alpha\Big] \\
&+ 3L^3\Big[\big(U\mathcal{E}U^{-1}\big)_{\alpha\beta} + \delta_{\alpha\beta}\mathcal{A}_\alpha\Big]\Big[\big(U\mathcal{E}^2U^{-1}\big)_{\beta\alpha} + (\mathcal{A}_\alpha + \mathcal{A}_\beta)\big(U\mathcal{E}U^{-1}\big)_{\beta\alpha} + \delta_{\alpha\beta}(\mathcal{A}_\alpha)^2\Big] \\
= &-3L^3\Big[\big(U\mathcal{E}^2U^{-1}\big)_{\alpha\beta}\big(U\mathcal{E}U^{-1}\big)_{\beta\alpha} - \big(U\mathcal{E}U^{-1}\big)_{\alpha\beta}\big(U\mathcal{E}^2U^{-1}\big)_{\beta\alpha}\Big],
\end{aligned}
\tag{17}
$$

where all the contributions of the matter effect have disappeared again in the last step. Since the last expression in (17) is the term cubic in $\Delta E_{jk}L$ for the probability in vacuum, we obtain

$$
\sum_{j,k} U^M_{\alpha j}U^{M*}_{\beta j}U^{M*}_{\alpha k}U^M_{\beta k}\big(\Delta E^M_{jk}L\big)^3 = \sum_{j,k} U_{\alpha j}U^*_{\beta j}U^*_{\alpha k}U_{\beta k}\big(\Delta E_{jk}L\big)^3. \tag{18}
$$

It turns out that the matter contributions in the terms of $\mathcal{O}((\Delta E_{jk}L)^4)$ or higher are not canceled and we have

$$
P(\nu_\alpha \to \nu_\beta)_{\mathrm{matter}} = P(\nu_\alpha \to \nu_\beta)_{\mathrm{vacuum}} + \mathcal{O}\big((\Delta E_{jk}L)^4\big). \tag{19}
$$

We note in passing that Eq. (18) gives another proof of the Harrison–Scott identity [12] for the case with three flavors [2]

$$
J^M \Delta E^M_{31}\Delta E^M_{32}\Delta E^M_{21} = J\Delta E_{31}\Delta E_{32}\Delta E_{21}, \tag{20}
$$

for

$$
\begin{aligned}
&\sum_{j,k} U^M_{\alpha j}U^{M*}_{\beta j}U^{M*}_{\alpha k}U^M_{\beta k}\big(\Delta E^M_{jk}\big)^3 \\
&\quad = i\sum_{j<k}\Im\big(U^M_{\alpha j}U^{M*}_{\beta j}U^{M*}_{\alpha k}U^M_{\beta k}\big)\big(\Delta E^M_{jk}\big)^3 = iJ^M\Big[-\big(\Delta E^M_{13}\big)^3 + \big(\Delta E^M_{23}\big)^3 + \big(\Delta E^M_{12}\big)^3\Big] \\
&\quad = -3iJ^M\Delta E^M_{31}\Delta E^M_{32}\Delta E^M_{21} = \sum_{j,k} U_{\alpha j}U^*_{\beta j}U^*_{\alpha k}U_{\beta k}(\Delta E_{jk})^3 = -3iJ\Delta E_{31}\Delta E_{32}\Delta E_{21},
\end{aligned}
\tag{21}
$$

where

$$
J^M \equiv \Im\big(U^M_{\alpha 1}U^{M*}_{\beta 1}U^{M*}_{\alpha 2}U^M_{\beta 2}\big), \tag{22}
$$

$$
J \equiv \Im\big(U_{\alpha 1}U^*_{\beta 1}U^*_{\alpha 2}U_{\beta 2}\big) \tag{23}
$$

are the Jarlskog factors in matter and in vacuum, respectively, and we have used the fact $a^3 + b^3 + c^3 = a^3 + b^3 - (a+b)^3 = -3ab(a+b) = 3abc$ for $a + b + c = 0$ ($a \equiv \Delta E_{13}$, $b \equiv \Delta E_{32}$, $c \equiv \Delta E_{21}$).

For long baseline experiments such as JHF [14] with relatively low energy ($E_\nu \sim 1$ GeV, $L \sim 300$ km), the larger mass squared difference $|\Delta m^2_{32}| \sim 3 \times 10^{-3}$ eV^2 gives $|\Delta m^2_{32}L/2E| \sim \mathcal{O}(1)$ and our assumption does not hold. In fact it has been shown [15] that there is some contribution from the matter effect to CP violation at the JHF neutrino experiment.

So far we have assumed that the density of matter is approximately constant. However, even if the density depends on the position, if adiabatic treatment is allowed (i.e., $|dU^M/dt| \ll |E^M_j|$) then we can apply our argument to each interval in which the density can be regarded as approximately constant. Hence, vacuum mimicking phenomena occur if adiabatic treatment is justified and $|\Delta E_{jk}L| \ll 1$ and $|A_jL| \ll 1$ are satisfied.

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[2] A different form of the quantity J^M/J has been given in [13].

Note added

After this paper was submitted to the preprint archive, the author has learned from E. Akhmedov that the result here holds not only in matter of approximately constant density, but also in the case of an arbitrary density profile [16].

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